Geometry and Tonal Music:
A Mathematical and Musical Analogy in Microtonal Systems

by

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Honors Thesis

Appalachian State University

Submitted to the Department of Mathematical Sciences
and The Honors College
in partial fulfillment of the requirements for the degree of

Bachelor of Science

May 2014

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Abstract

Using Brian J. McCartins "Prelude to Musical Geometry" as a guide, we will look at the geometric link between mathematics and tonal music. When a collection of pitches is heard, either successively (melodically) or simultaneously (harmonically), forming scales or chords, we find varying intervallic relationships. One of our main goals in investigating this musical link is to understand the mathematical implications of the interval relationships among the members of any set of pitches and systems. Arranging the pitches of an n-tone system in a circle of pitches similar to a clock, we are able to plot the pitches of a chord on our musical clock to see the geometric distances, or semitones, between each note. These pitches are joined together by line segments, forming a polygon that we then try to rotate (transpose) and/or reflect (invert). From what is known about the 12-tone system, we hope to see the same geometric and mathematical representation in a 20-tone and then other general n-tone systems. We ultimately hope to find analogies between different n-tone systems the applications of the same geometric and mathematical representations in order to see which, if any, microtonal systems preserve the same chordal structure and properties present in the 12-tone system that makes up the musical world we know.

Contents

1 Introduction

"Musical form is close to mathematics–not perhaps to mathematics itself, but certainly to something like mathematical thinking and relationship."

–*Igor Stravinsky, 1882-1971* [1]

From the time Pythagoras investigated the laws of the vibrating string in sixth century BC, mathematicians have been fascinated by the subtle relations, analogies, and connections that exist between music and math [1]. While the ancient Chinese, Egyptians and Mesopotamians are know for their studies of the mathematical principles of sound, the Pythagoreans of ancient Greece were the first researchers to investigate the expression of musical scales in terms of numerical ratios [2], [3]. It's believed that Pythagoreans' central doctrine was that "all nature consists of harmony arising out of numbers" [4].

Harmonic and rhythmic music continued to be investigated through physics and mathematics as theorists and mathematicians tried to bridge the gap between the two related fields. As the relationship between composing and hearing music has been investigated by mathematicians and musicians alike, musical applications of set theory, abstract algebra, number theory, the golden ratio, and Fibonacci numbers have occurred. With the creation of the computer and its many capabilities, new doors of modeling and making music were opened as was the possibility of exploring alternative microtonal systems of octave division [5].

1.1 Microtonal Music

Microtonal music as a whole has a broad history, and these n-tone systems and their "rules" are distinct to the culture and country from which they come. It is well-known and accepted that most of the musical instruments of the western world's orchestra are constrained to produce 12 distinct pitch classes, with each pitch occurring once per octave. While Western music is based on this 12-tone (12 equal temperament) system, other cultures developed different systems; Persian music divides octaves into 24 quarter tones (24 equal temperament), Arabic music uses a 16-tone system (16 equal temperament), and the Chinese use a different form of the 12-tone system (12 equal temperament) [6].

The 12-tone system which makes up the western musical world we know consists of 12 "equally spaced" pitches

$$\{C,\ C^\sharp/D^\flat,\ D,\ D^\sharp/E^\flat,\ E,\ F,\ F^\sharp/G^\flat,\ G,\ G^\sharp/A^\flat,\ A,\ A^\sharp/B^\flat,\ B\}.$$

We number these pitches as $0, 1, 2, 3, ..., 11$ in order to model the 12-tone system and as a means of organizing pitch relationships. "The basic principle of the keyboard is that any key that is twelve half steps [the interval between any two adjacent keys] above another key produces a pitch whose frequency is exactly double that of the lower key. To the ear, these two sounds seem related, and they are defined as being one *octave* apart" [7]. On a keyboard, each new octave begins with 0 (or C) and, in our model, is mathematically the same as any other C that could be played. However, musically, middle C does not have the same frequency as high C, low C, or any C in between, but the sounds are highly related and to simplify our model, we will call all C's the same. We treat any two notes that differ by an octave as equivalent, as the corresponding numbers are equivalent and the notes (while having distinct pitches) sound enough alike for us to consider them in the same "pitch class." In the 12-tone system, seven of the pitches, $\{A, B, C, D, E, F, G\}$, are designated as natural notes, represented by the white keys on the keyboard of a piano or organ. The remaining five notes make up the black keys on the keyboard, which musically are our sharps and flats $\{C^\sharp/D^\flat,\ D^\sharp/E^\flat,\ F^\sharp/G^\flat,\ G^\sharp/A^\flat,\ A^\sharp/B^\flat\}$.

Piano keyboards–our musical instrument of choice–are typically tuned in equal temperament where any two adjacent keys, regardless of whether they are a white or black key, are at the same musical distance from one another and this distance, or interval, between each note is called a half-step, or semitone [8]. Thus, the distance between each white and black key on our keyboard, as well as the distance between E (4) and F (5),

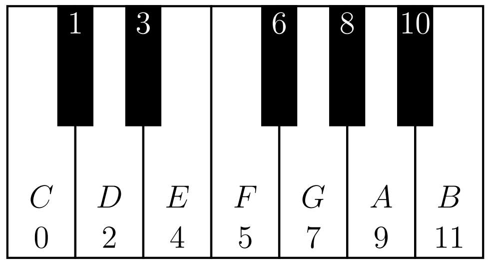


Figure 1: The 12-tone system on a piano keyboard.

is the same; this distance being equal to one semitone apart.

These 12 pitches or tones can be arranged in a circle of pitches similar to a clock, with the pitch C being labeled 0 instead of 12. We use the group $\mathbb{Z}_{12}$ as our model for the pitch classes so adding semitones to our musical clock makes use of arithmetic *modulo* 12 (mod 12). The Cayley Diagram of $\mathbb{Z}_{12}$ with generator 1 gives a visual model of the pitch-classes and their relationships. The diagram, or clock, is connected by stepping 1 semitone around the entire clock and then mod 12. Using this musical clock, we are able to see the geometric representations of chords, scales, et cetera and visualize the musical distances, or semitones, between each note.

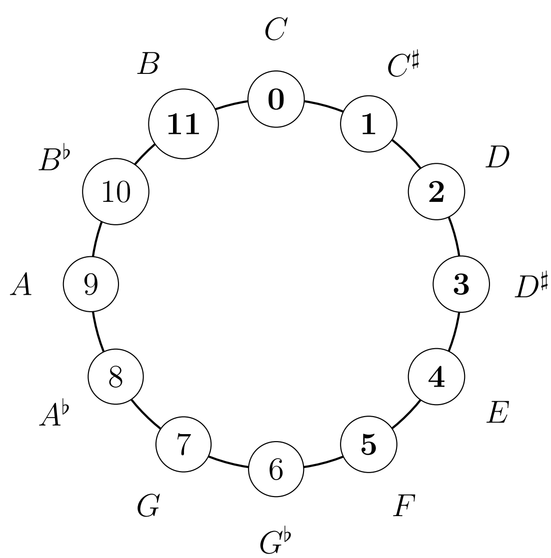


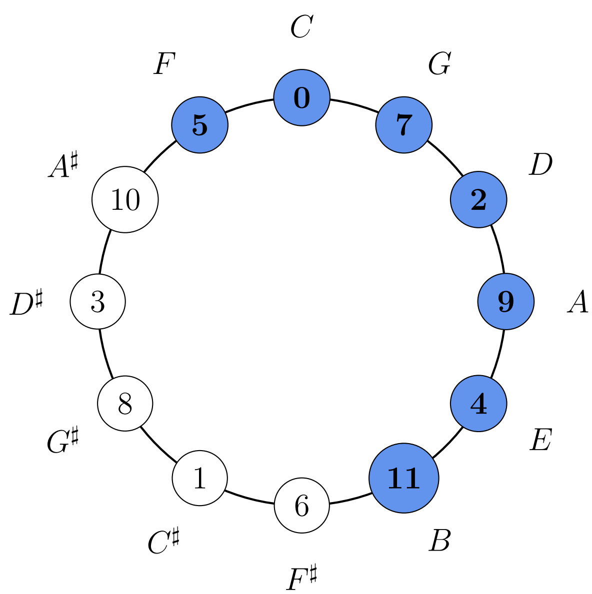
Figure 2: 12-tone musical clock

2 Twelve vs. Twenty: Diatonic Scale

One of the challenges we faced when looking into other microtonal systems was whether the structure, behaviors, and properties of the 12-tone system could hold true in other n-tone microtonal systems, such as $\mathbb{Z}_{20}$.

2.1 Circle of Fifths

To people with any music theory background, the circle of fifths, depicted below, is a familiar image.



$\mathbb{Z}_{12} = \langle 7 \rangle$

Figure 3: Circle of Fifths for 12-tone system.

The circle of fifths derives its name from the fact that, if we proceed clockwise around the clock, each successive key is obtained by transposing the previous key up by a perfect fifth [8]. Musicians refer to a seven-semitone step, the musical interval between notes consisting of five lines or spaces on the musical staff, as a *fifth*. As you move clockwise within the circle of fifths, you add 7 semitones and each point on the clock is 7 semitones apart from its neighboring note. Thus in the circle of fifths we are viewing, $\mathbb{Z}_{12}$ as the cyclic group generated by 7 mod 12, and the circle of fourths is generated by 5, which can be seen if you move counterclockwise on figure 3 above. In music theory, a (standard) diatonic scale is a seven-note scale comprising of two types of intervals–whole steps and half steps, with the half steps maximally separated [9]. This connects mathematically to the idea that scales are obtained from a root and chain of six successive fifths rescaled in an octave [9], i.e. a connected region of 7 in the Cayley graph of $\mathbb{Z}_{12} = \langle 7 \rangle$. Highlighted in figure 3 above is the C major diatonic scale, the key signature that has no flats or sharps. If we wanted to transpose or go to a G major, we would simply rotate the blue shading on the above figure clockwise by one fifth, dropping the F and picking up the $F^{\sharp}$.

Additionally, an important property of the 12-tone system is that it maintains an F to $F^{\sharp}$ property, which means that every time we move clockwise one step around the circle of fifths, we drop a note and the note picked up is exactly one semitone higher than the one we dropped [5]. This implies closeness in key rather than pitches when moving around the circle, thus grouping the white (natural) and black (sharp and flat) keys together in interconnected regions. The C-major diatonic scale tells us exactly which notes are the white keys and which are the black. Highlighted in figure 3, the blue shaded region in the $\mathbb{Z}_{12}$ circle of fifths represents the white keys on the keyboard, a connected region of 7, and the white region of the circle represents the black keys. Thus, to introduce a new keyboard for an n-tone system, we need to find the C-major diatonic scale.

When creating an n-tone keyboard for $n \geq 12$ that has musical properties similar to the 12-tone Western music system, we have decided to require

1. a tritone, i.e. a half-way point or note. This implies that n must be even.
2. that the number of white keys, i.e. the number of notes in our diatonic scales must be greater than and as close to half the number of notes as

possible (we would like to have more white keys than black).

3. the diatonic scales possess the F to $F^\sharp$ property.

These musical considerations lead to the following mathematical requirements listed below.

1. If we have a tritone, we know n is even, so $n = 2t$ for some integer $t \geq 6$.

2. Let k be the number of notes in our diatonic scale the $\mathbb{Z}_n = \langle k \rangle$. This is only possible if $gcd(n, k) = 1$. If n is a multiple of 4 ($n = 4t$ for some $t \geq 3$), we can have exactly one more white key than black ($k = 2t + 1$) because

$$\begin{aligned} 1 &= (4t)(t) + (2t+1)(-2t+1) \\ &= gcd(4t, 2t+1). \end{aligned}$$

If $n = 4t + 2$, we cannot have exactly one more white key than half because $\frac{4t+2}{2} + 1 = 2t + 2$ which is clearly not relatively prime to $4t+2$. However, we can have two more than half the keys be white because if $k = \frac{4t+2}{2} + 2 = 2t + 3$,

$$\begin{aligned} gcd(n, k) &= gcd(4t+2, 2t+3) \\ &= gcd(2t+1, 2t+3), \text{ and} \\ 2 &= (2t+3)(1) + (2t+1)(-1) \\ gcd(2t+1, 2t+3) &\leq 2 \\ gcd(2t+1, 2t+3) &\neq 2, \text{(because } 2t+3 \text{ is odd)} \\ gcd(4t+2, 2t+3) &= gcd(2t+1, 2t+3) = 1 \end{aligned}$$

Therefore, if $n = 4t$, then $k = 2t + 1$ and if $n = 4t + 2$, $k = 2t + 3$.

3. If our n-tone, $\mathbb{Z}_n = k$ system possesses the F to $F^\sharp$ property, we require $k^2 - 0 = 1$ as illustrated in the following diagram.

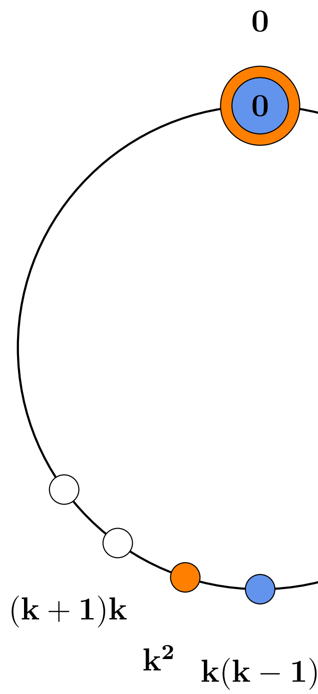


Figure 4: F to $F^\sharp$ Property: $k^2 - 0 = 1 \mod n$

If $n = 4t$, $k = 2t+1$ and the F to $F^\sharp$ property holds true as $k^2 = (2t+1)^2 \mod 4t = 4t^2 + 4t + 1 \mod 4t = 1$. However, if $n = 4t+2$, so $k = 2t+3$, the F to $F^\sharp$ property is not possessed.

$$\begin{aligned}
k^2 \mod 4t+2 &= (2t+3)^2 \mod 4t+2 \\
&= 4t^2 + 6t + 9 \mod 4t+2 \\
&= t(4t+2) + 2(4t+2) + (2t+5) \mod 4t+2 \\
&= 2t+5 \mod 4t+2 \\
&= 2t+5 \text{ (because } t > 1) \\
&\neq 1
\end{aligned}$$

These mathematical consequences lead to the following theorem.

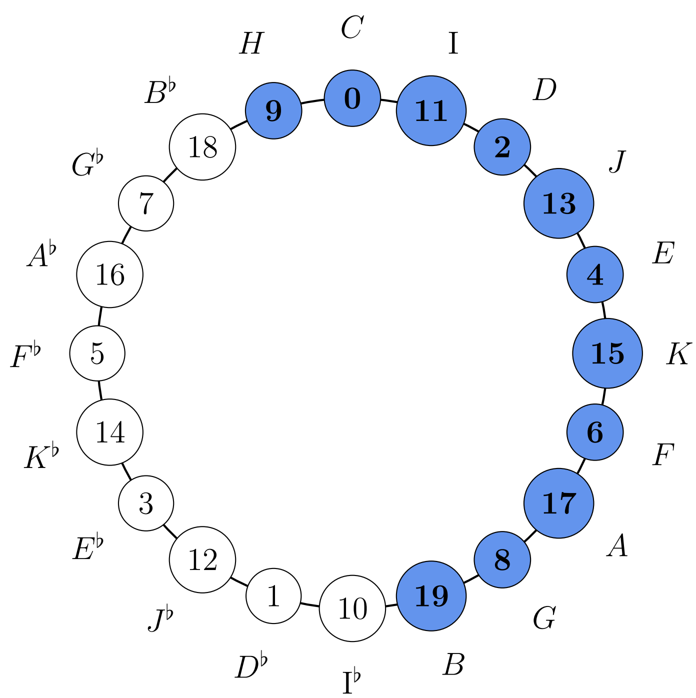
Theorem 1. *In a microtonal musical system with an even number of semitones, n with diatonic scales consisting of $k > \frac{n}{2}$ semitones, if $k - \frac{n}{2}$ is at a minimum and the diatonic scales possess the F to $F^\sharp$ property, then $n = 4t$ for some integer t and $k = 2t + 1$.*

Thus for the 12-tone system, our theorem says that $k = 7$, which gives us the diatonic scale we are familiar with. For a 16-tone system, we would chose a 9 note diatonic scale, and for a 20-tone system, we would choose an 11 note diatonic scale, and so on and so forth. However, we will show later that for a 16-tone system, the chordal structure is dissimilar from $\mathbb{Z}_{12}$, thus we begin looking at a 20-tone system.

2.1.1 Balzano and Zweifel and 20-tone "Circle of Fifths"

Balzano and Zweifel, two musicians who were particularly interested in microtonal systems expanded what was known about the 12-tone system in regards to a 20-tone and other n-tone systems. While both discussed the case of $\mathbb{Z}_{20}$ in a similar nature, each musician made different decisions on how to define the diatonic scale for this system. Balzano believed the diatonic scale should be made up of 9 notes, while Zweifel chose an 11-tone scale. Both Balzano and Zweifel made solid arguments and cases for their scale choices, but it seems that an 11-tone scale makes more sense when looking at parallels between $\mathbb{Z}_{12}$ and $\mathbb{Z}_{20}$, as indicated in our previous discussion. Choosing 11 as our generalized circle of fifths, or more specifically our circle of ninths generator, leaves 9 to be our other complimenting generator since $-11 = 9$ in $\mathbb{Z}_{20}$. $\langle 1, 3, 9, 11, 17, 19 \rangle$ generate $\mathbb{Z}_{20}$. Zweifel was able to form a generalized circle of fifths for the 20-tone system by taking a connected region of 11 starting two rotations back from C and calling that the C-Major scale. Our system is generated using a modification of Zweifel's 11-tone diatonic scale as seen in the figure below.

Using this C-major diatonic scale, we are able to create a 20-tone keyboard. This keyboard has the same look and "feel" of the 12-tone standard keyboard, with adjacent



$$\mathbb{Z}_{20} = \langle 9 \rangle =$$

Figure 5: Circle of Ninths for 20-tone system.

white keys occurring in structurally similar places from the modified diatonic scale Taylor Askew developed [11].

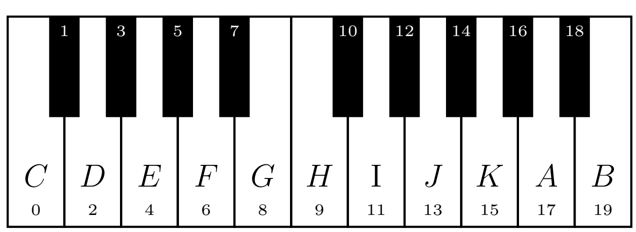


Figure 6: 20-tone Keyboard.

The 20-tone system has eleven white keys and nine black keys (complimenting the generators 9 and 11) similar to how the 12-tone system (figure 1) has seven white keys and 5 black keys (with generators 5 and 7) [10].

3 Chord Structure

Arranging the pitches of an n-tone system in a circle of pitches similar to a clock, we are able to plot the pitches of a chord on our musical clock to see the geometric distances, or number of semitones, between each note in the chord. These pitches are joined together with line segments, forming a polygon, a visual representation of any given chord.

A 12-tone major scale is made up of a major third (4 semitones) followed by a minor third (3 semitones). These step-sizes have mathematical significance, as $4 + 3 = 7$, our generator for the circle of fifths. Splitting the 12-tone diatonic scale into two pieces, we get pieces of length 4 and 3–the major and minor third. The remaining length of the polygon is made up by the number of semitones equivalent to the complimenting generator; for the 12-tone system, this means the final piece of the polygon is a separation 5 semitones. We can see this chord representation in figure 7 below.

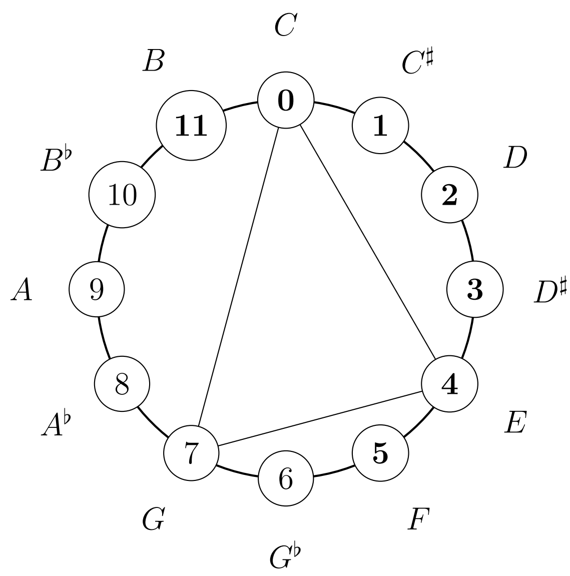


Figure 7: C Major Chord in 12-tone system

C and E are 4 semitones apart, E and G are separated by 3 semitones, and the distance from G to the next C (given that all notes one octave apart are mathematically equivalent) is a length of 5 semitones.

For the 20-tone system, we wanted to be able to see and represent the same thing, splitting the 11 generator into pieces of length 6 and 5. These two lengths form our generalized major and minor "thirds," or more specifically our major and minor fourths, in our 20-tone system.

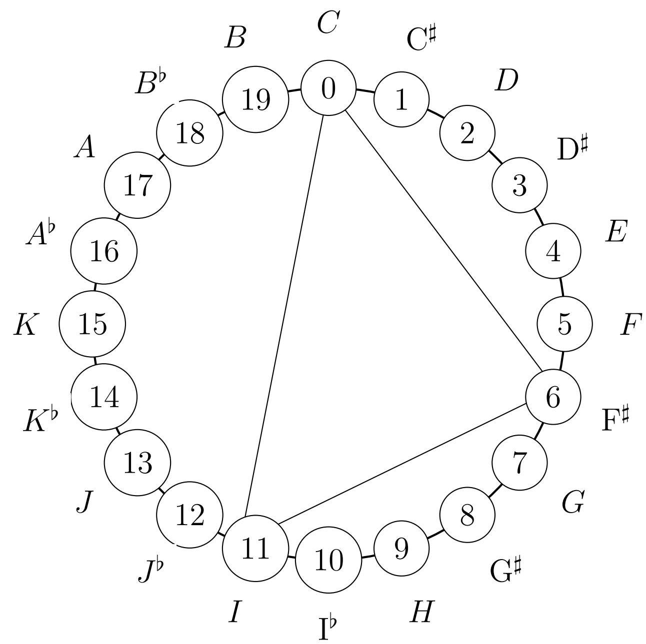
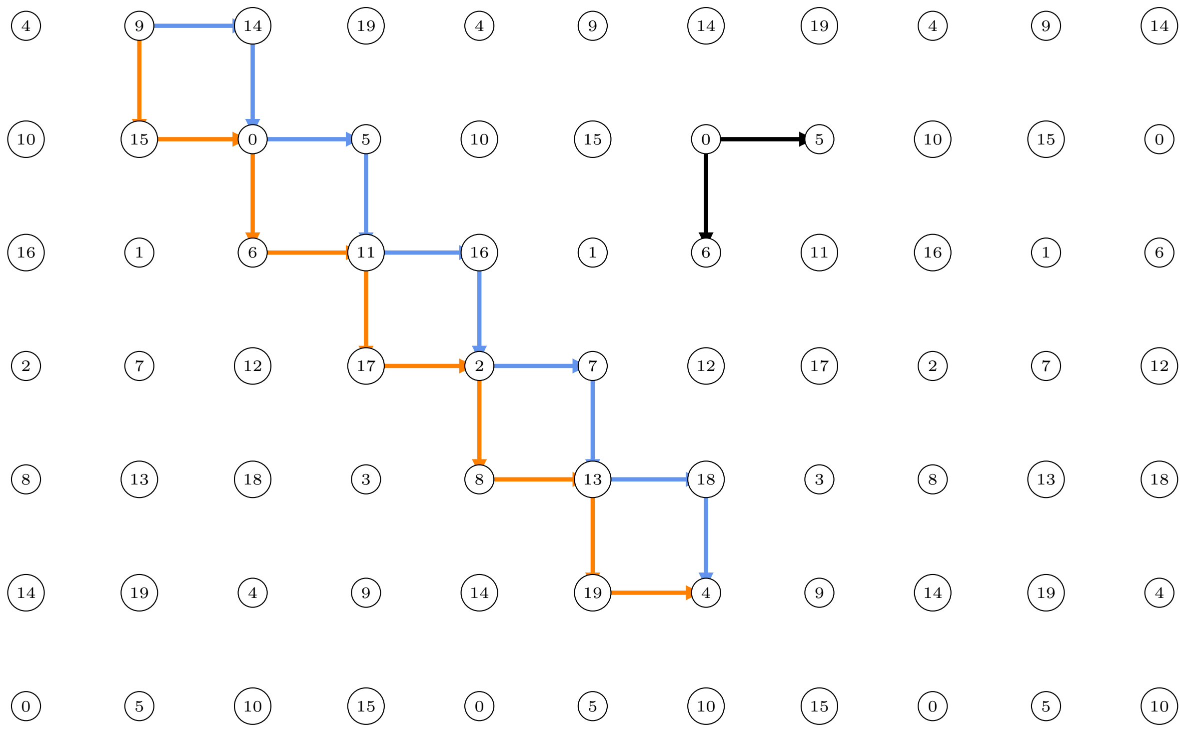


Figure 8: C Major Chord in 20-tone system

Like the 12-tone representation of the C Major chord, we see that the lengths between *C* and *E* are the longer piece length, separated by 6 semitones, and *E* and *G* are 5 semitones apart; these two lengths sum to 11, the circle of ninths generator for this system. The distance between *G* and *C* is 9 semitones, the complimentary 20-tone generator. By mathematically rotating and reflecting–or musically transposing and translating–we are able to generate all possible major (rotations only) and minor (reflection followed by rotation) chords for a system. This rotation and reflection property holds true for both microtonal systems.

3.1 Cayley Diagrams with Multiple Generators

Another way to interpret major and minor chords are Cayley diagrams in which we choose multiple generators, one for the major step and one for the minor. Through these diagrams, we are able to see all the notes as they appear in C-major diatonic (white notes) scale. Furthermore, defining the diatonic scale in this 11-tone way distinguishes our chord structures and represented Cayley diagrams as distinct and different than those chord diagrams generated by Balzano and Zweifel. For $\mathbb{Z}_{20}$, our major triad, or "third," can be defined by the root, the note 6 semitones following that, then the note 5 semitones following that note [11]. The minor triad is the inversion of the major, meaning it is comprised of the root, 5 semitones following, and 6 semitones following that. Balzano and Zweifel both represented $\mathbb{Z}_{20}$ in a Cayley Diagram generated by the triadic steps. Our Cayley Diagram is different than previous ones because we made a major triad a M4 (Major Fourth made up of 6 semitones) and a m4 (minor fourth made up of 5 semitones), which parallels the 12-tone system's major triad of M3 (Major Third of 4 semitones) and m3 (minor third of 3 semitones) [11]. This means that the C-Major tonic triad is 0, 6, 11. Because of this method of defining our triads, our generators for $\mathbb{Z}_{20}$ are 5 and 6. A Balzano diagram of this group would be generated by 4 and 5 since the direct product $(4 * 5)$ makes up the group (20), but since 5 and 6 are relatively prime to 20, they still generate all of $\mathbb{Z}_{20}$. The only difference between our $\mathbb{Z}_{20}$ graph (generated by $\mathbb{Z}_{20} = \langle 5, 6 \rangle$) and Balzano's graph generated by $\mathbb{Z}_{20} = \langle 4, 5 \rangle$ is that rather than having this be a nice unfolded torus, figure 9 is an unfolded twisted torus–it's more of a Krisy Kreme cruller than the hot-and-fresh standard glazed doughnut torus. Every 0 in the graph is the same one, but in order to "glue" the $(0's, 1's, ..., 14's)$ back together to form a torus, it requires a twist to adapt to the change in generators [11]. We chose to create our own Cayley diagram rather than continue with Balzano's graph because our diatonic scale more closely mirrors the properties of the $\mathbb{Z}_{12}$ diatonic scale as Zweifel showed that the nine-tone $\mathbb{Z}_{20}$ scale presented by Balzano is more like the $\mathbb{Z}_{12}$ pentatonic scale (the scale produced by generator 3 in the 12-tone system or 9 in the 20-tone system) in structure [10].

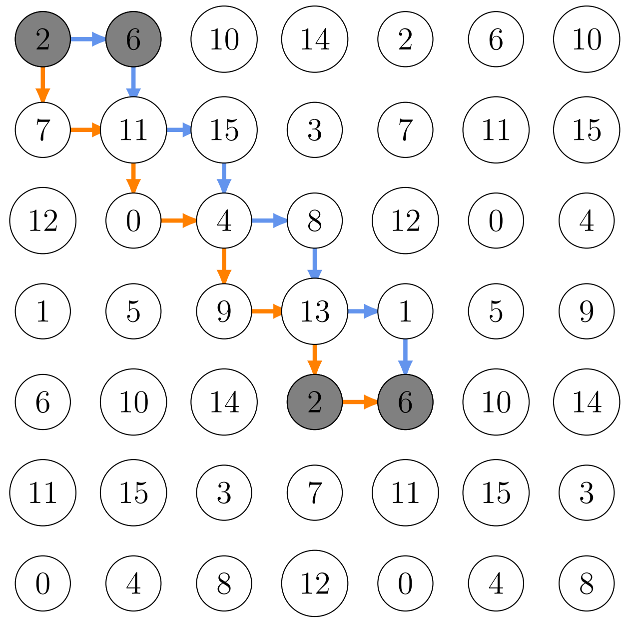


$$\mathbb{Z}_{20} = \langle 5, 6 \rangle$$

Figure 9: $\mathbb{Z}_{20}$ Torus with C Major/Minor Chords

The above Cayley Diagram provides us with a nice visual way to see how these major and minor chords relate. As we were able to see in the circle of fifths transform for $\mathbb{Z}_{20}$ (figure 5), a major scale is any connected region of eleven. We can see that C Major $(0, 6, 11)$ is highlighted in orange and generated by taking 6 steps (or semitones) down and then an additional step of size 5. What's more, we can see that we are able to get every note (or number) present in the C-major diatonic scale (i.e. every natural key) by following the major chords in the diagram without repeating any notes by taking steps of size 6 then 5. This saw-toothed process of stepping shows the full process so that we could glue the ends back together three-dimensionally (0 to 0, 14 to 14, *etcetera*) in order to form a doughnut, or torus. However, for our $\mathbb{Z}_{20}$ graph, this torus is twisted. Likewise, all notes are passed through and minor chords are generated by taking the inverse of the steps, 5 then 6. C Minor $(0, 5, 11)$, highlighted in blue, is generated by taking a step of size 5 followed by an additional step of 6 semitones.

As mentioned earlier, the $\mathbb{Z}_{16}$ system does uphold the chordal structure of both the 12-tone system and our 20-tone system. As illustrated in the Cayley Diagram below (figure 10), the path the major chords trace on the twisted torus intersects itself, which means the structure would fall back on itself before encompassing all of the natural notes, forming a pretzel of sorts. We can see where issues arise with figure 10's highlighted 2 and 6. As we can see in the $\mathbb{Z}_{16}$ major chord progression (highlighted in orange), 2 is not the end of any chord, yet it is still repeated like it is. This repeat leaves two chords that should be very different relatively similar as they share the same note 2, causing the chord path to intersect itself. The same can be said with the 6 note in regards to $\mathbb{Z}_{16}$ minor chords (highlighted in blue). Because major and minor chords are key in Western music composition, this difference was unacceptable to us.



$\mathbb{Z}_{16} = \langle 4, 5 \rangle$

Figure 10: $\mathbb{Z}_{16}$ Torus with C Major/Minor Chords

We started with 20-tones because it had been investigated somewhat by previous scholarly research and held the same mathematical and musical qualities, properties, and similarities we can see with the 12-tone system. We knew that we needed an even system so we could maintain the tritone element, thus our n required a halfway point. Additionally, we knew the "circle of fifths" generator should be larger than that halfway point and as close as possible to the tritone. When we began comparing chord structures on the 12- and 20-tone systems, we found similarities in shape, semitone distance, generators, and Cayley Diagrams. Knowing all of these things mirrored what was seen in the 12-tone system, we were hopeful our 20-tone system could uphold and demonstrate further similarities and properties as we continued our investigation into the mathematical and musical analogy in microtonal systems between geometry and tonal music.

4 Interval Vectors and All-Interval Chords

As explained by Brian McCartin [12], a chord to contain n distinct pitches and thus have

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

pairs of pitches. These n distinct pitches are played simultaneously to produce a chord. As we investigated earlier when looking at similarities between chord structure in 12- and 20-tone systems, "each pair can be pictured as an edge or a diagonal of the chord polygon, and the interval between them is the number of hours, from 1 to 6, separating them on the musical clock [12]." 6 was chosen for the 12-tone system because within the chord representation on the 12-tone musical clock, the separation of each pair of notes in the chord can have up to 6 intervals (since 6 pairs of pitches = 12 tones and 6 is the maximal distance separating any two tones in the clock). This can be mimicked for the 20-tone system as we'll later see.

Knowing this, we can create a vector, the chord's *interval vector*, whose entries record how many pairs of pitches in the chord are separated by 1 semitone, how many by 2 semitones, et cetera. The most important thing to take away from this is that chord's interval vector reveals how many pitches will be unchanged by a given transposition [12]. Looking back at figure 7, we see that the interval vector for the C major triad is $\langle 0, 0, 1, 1, 1, 0 \rangle$, since this chord has one pair of pitches separated by each of 3, 4, and 5 semitones, and no pairs at other intervals [12]. It should be noted that these semitone separations can be both internal (as shown by the dashed lines in figure 11 to come)–those distances within the circle connecting two pitches not adjacent to each other–and external (as shown by the solid lines in figure 11 to come)–the interval distances created by the tones and their external placement on the musical clock.

4.1 All-Interval Chords

An all-interval chord is a chord containing six pairs of pitches separated by the six different intervals, so with the interval vector $\langle 1,1,1,1,1,1 \rangle$ [12]. To be an all-interval chord, all possible interval distances must be represented and pairs are the same interval distance apart. In the 12-tone system this is represented by all-interval chords with two pitches that are 1 semitone apart, two pitches that are 2 semitones apart, and so on and so forth to two pitches that are 6 semitones apart.

Using a little algebra, we can determine that the number of distinct pitches in a 12-tone all-interval chord from McCartin's chord definition/equation.

$$\begin{aligned} 6 &= \frac{(n)(n-1)}{2} \\ 12 &= n^2 - n \\ 0 &= n^2 - n - 12 \\ 0 &= (n+3)(n-4) \end{aligned}$$

We determine that the number of notes in a 12-tone all-interval chord can either be -3 or 4, thus the solution to this equation and number of pitches in an all-interval chord is $n = 4$. This can be checked by

$$\binom{4}{2} = \frac{4(4-1)}{2} = \frac{4(3)}{2} = \frac{12}{2} = 6$$

thus the 6 pairs of pitches in the chord all are separated by different intervals and our chords can contain up to 4 tones. Using this same logic and order, we test this for the 1 to 10 interval in the 20-tone system, knowing we need to end up with 10 pairs of pitches if we are to have an all-interval chord with interval vector $\langle 1,1,1,1,1,1,1,1,1,1 \rangle$.

$$\begin{aligned} 10 &= \frac{n(n-1)}{2} \\ 20 &= n^2 - n \\ 0 &= n^2 - n - 20 \\ 0 &= (n+4)(n-5) \end{aligned}$$

So if this equation and logic holds true, we should see all-interval chords containing five pitches in a 20-tone system.

In *Prelude to Musical Geometry* [12], Brian McCartin offers a detailed geometric argument showing that up to transposition/inversion, the only all-interval chords in the 12-tone system are the tetrachords $P_1 = \{B, C, D, F^\sharp\}$ and $P_2 = \{C, C^\sharp, E, F^\sharp\}$ as seen in figure 11.

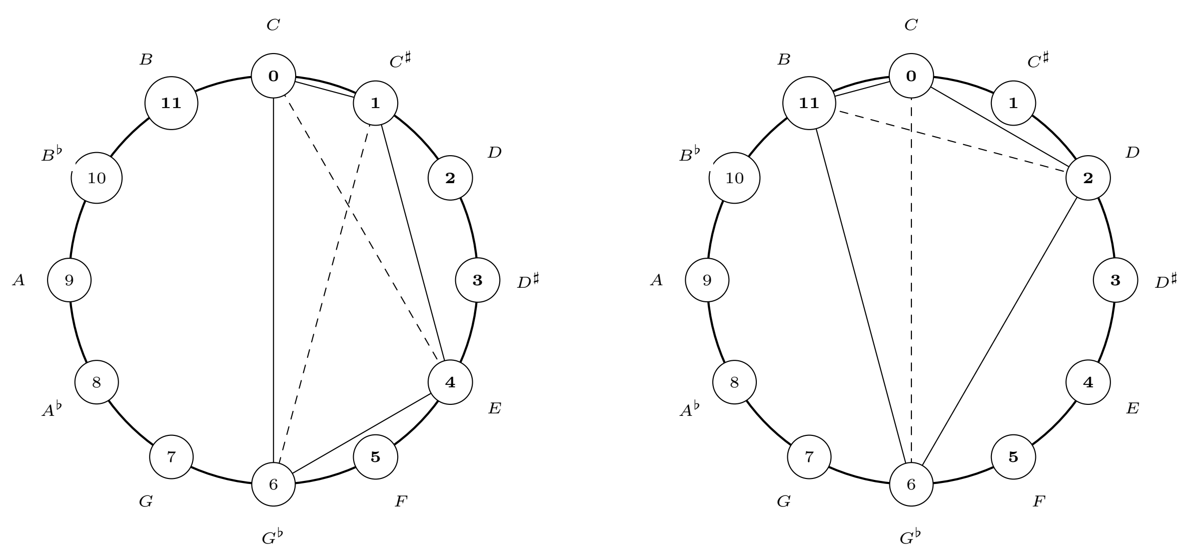


Figure 11: All-interval chords in the 12-tone system

Given the many parallels of the 12-tone system we've already seen mirrored in our 20-tone system, we were hopeful that we could extend McCartin's geometric proof and find and list the all-interval chords for a 20-tone system. However, we actually found that no such all-interval chords existed.

Theorem 2. *There are no all-interval chords in the 20-tone system.*

Proof. Let P be any chord with interval vector $\langle 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \rangle$. Thus, P contains one pair of pitches separated by a "tritone" of our 20-tone system; this length-10 "tritone" may appear as either a diagonal or an edge of the associated pitch polygon. From this interval vector, we know that all lengths are distinct and present in the pentagon, meaning we need edges of length $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. The external edges make a simple closed path inside the circle. Since the circle is of length 20, the path must be as well, thus the sides must sum to 20.

In determining the sides, 10 has to be either interior or exterior. Suppose first length 10 appears as an exterior length of P. By rotating the clock face, we can arrange this length to join positions 0 and 10. Because 10 is exterior, the remaining four sides must sum to 10 to close the polygon. But the only possible partition of 10 that fulfills these claims and splits the partition into four distinct terms is $\{1, 2, 3, 4\}$. However, this cannot occur as placing these lengths next to each other will result in multiple internal lengths of the same distance: If 1 was placed next to 2 (as shown by the dashed line in figure 12 below), there would be an internal length of 3. Similarly, 1 cannot be placed next to 3 as there would now be an interior length of 4. Thus, 1 must be placed next to 4, but then 2 and 3 are also adjacent, leaving two internal lengths of 5.

Now we know the length 10 must be interior as shown in the figures below.

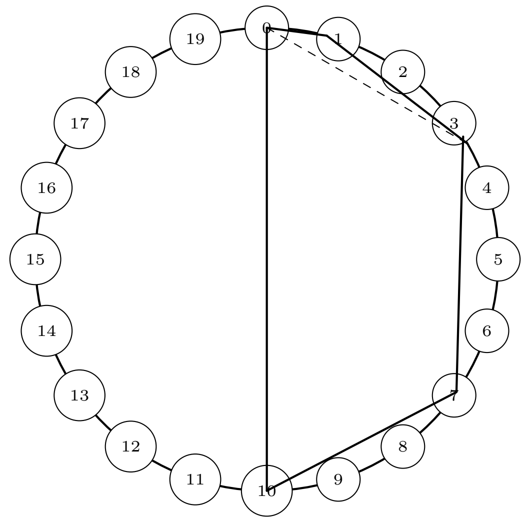
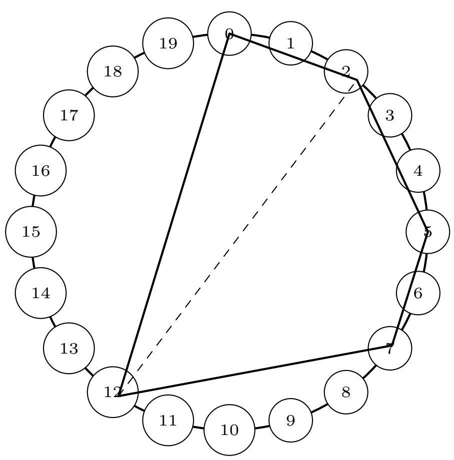
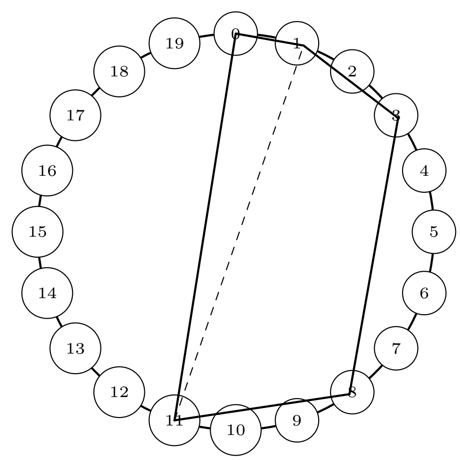


Figure 12: Interior Length-10

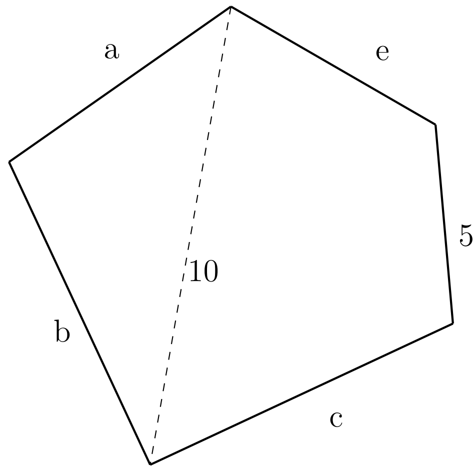
The length 5 edge cannot be adjacent to the length 10 edge because this would create a second length 5 edge joining the two.





Suppose that 5 is an external edge. Because it cannot be adjacent to the length 10 edge, it must appear as in figure 13.

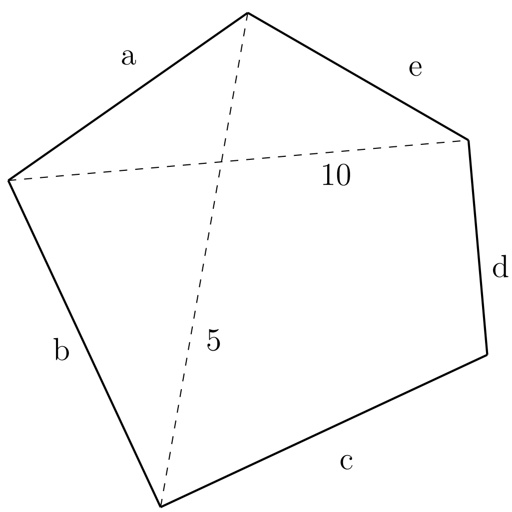
If $a = 1$ and $b = 9$ ($\{1, 9\}$), then $\{c, e\} = \{2, 3\}$. However, $e \neq 2$ because the internal length $a + e + 5 = 8$ is equivalent to another repeating internal length of 8 ($c + 5 = 8$). Additionally $e \neq 3$ because $a + e + 5 = 9 = b$. We can rule out all the other possible partitions in a similar way. Therefore, the length 5 edge must be interior.



$$
\begin{aligned}
a+b &= 10 \\
\{a,b\} &= \{1,9\},\{2,8\},\{3,7\},\{4,6\} \\
c+e &= 5 \\
\{c,e\} &= \{1,4\},\{2,3\}
\end{aligned}
$$

Figure 13: Length 10 interior, length 5 exterior

Since the length 5 edge cannot be exterior, it must be interior. In addition, because the length 5 edge cannot be adjacent to the length 10 edge, the length 5 edge must intersect the length 10 edge as shown in figure 14.



$$
\begin{aligned}
a+b+c+d+e &= 20 \\
a+b &= 5 \\
a+e &= 10
\end{aligned}
$$

Figure 14: Lengths 10 and 5 intersect

Thus, $\{a,e\}$ is one of our partitions summing to 10. This means $\{c,d\}$ must partition $10-b$, where b is completely determined by our choice of a and e.

For example, look at $\{a, e\} = \{2, 8\}$, so $b = 3$ and

$$\begin{aligned} c + d &= 7 \\ \{c, d\} &= \{1, 6\}, \{2, 5\}, \{3, 4\} \end{aligned}$$

For the partition $\{c, d\} = \{1, 6\}$, if $c = 1$ and $d = 6$, then $d + e = 6 + 8 = 14$ so the interior length $20 - 14 = 6$, but $d = 6$. If $c = 6$ and $d = 1$, then $d + e = 9$ and $c + b = 9$. The same logic can be applied for the other possible combinations partitions of 10 listed in figure 13.

Thus, after trying all possible arrangements of our pentagon, we found that there are ***no*** all-interval vectors, $\langle 1, 1, 1, 1, 1, 1, 1, 1, 1, 1 \rangle$, for the 20-tone system. $\square$

5 Conclusion

For $\mathbb{Z}_{20}$, we found many parallels between the 12- and 20-tone systems. Similarities occurred in (diatonic) chord diagrams, upholding the tritone and F to $F^\sharp$ properties, plotting major and minor chords on our musical clocks and Cayley Diagrams, and the presence of a doughnut-like torus on the $\mathbb{Z}_{20}$ torus (figure 9). However, when investigating interval vectors and all-interval chords in the 20-tone system, we found that there are no all-interval chords for $\mathbb{Z}_{20}$. This difference between $\mathbb{Z}_{12}$ and $\mathbb{Z}_{20}$ is the first demonstrated difference between our two systems, which raises the question of is the all-interval chord theorem always going to hold true for other microtonal systems? Now we're looking into generalizing what we've found in the 12- and 20-tone systems for other n-tone systems to see if all-interval chords exist in these microtonal systems.

While the majority of our research focused on diatonic scales (namely C-major), it would be interesting to try to find similarites and differences between our two systems–

$\mathbb{Z}_{12}$ and $\mathbb{Z}_{20}$–with minor and diminished chords. Once we have that knowledge, we should be able to investigate and find the similarities that exist between minor, diminished, and other chords across various microtonal systems.

Lastly, a major question that has been raised in numerous papers and scholarly research surrounds $\mathbb{Z}_{16}$ and why it doesn't "work." Is there something about $n = 12$ and $n = 20$ that make their chord structures so alike that doesn't ring true for odd toned systems or $n = 16$? Does this trend continue for other n-tone systems and are there specific conditions for generalizing these microtonal systems? Why doesn't $n = 16$ work and is there something significant about it crossing that causes the discontinuity? Are these differences in major and minor chord structure as represented in the Cayley diagrams always going to cause a difference in torus and system structure? What about the structure of $\mathbb{Z}_{16}$ makes its torus resemble a pretzel rather than having more of a doughnut shape? Many people, including ourselves, have voiced both similarities and differences between $\mathbb{Z}_{16}$ and other systems, but no real conclusions have come about beyond the difference in chord structure and the presence of a cross (figure 10).

6 Acknowledgements and References

Major thanks are due to Dr. Vicky Klima for all of her help throughout the past two semesters with this project. Thank you for pushing me and challenging me to think big and reach for the moon when it came to working towards my presentation at the National Conference of Undergraduate Research and for your careful guidance and leadership with my thesis and research. Thank you for putting up with my hectic schedule and irregular pace and for asking, posing and promoting "what if" questions that were hard to answer but helped me learn more in the long-run. I truly appreciate everything you've done to help me this year and for helping my research flourish as it did. I also thank Dr. Fankhauser for providing Dr. Klima and I with provide musical feedback and expertise when it came down to writing and supplementary research as my second reader. Appreciation is also due to continuous support my friends, family, coworkers, peers, and residents have shown me as I have worked on first research and then writing. Thank you for your diligence, encouragement, and understanding when I started ranting and raving about all things microtonal music and math.

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